A solution for three-dimensional vortex flows with strong circulation

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The Navier–Stokes equations for a viscous, incompressible fluid are considered for a steady, axisymmetric flow composed of a strong rotation combined with radial sink flow which exhausts axially inside a finite radius. The equations are reduced to two coupled partial differential equations in terms of the stream function and circulation. The equations contain three dimensionless parameters: the radial Reynolds number, a characteristic ratio of mass flow per unit length to circulation, and a characteristic ratio of an axial dimension to a radial dimension. The product of these last two dimensionless parameters is used as a new expansion parameter for generating an asymptotic series solution. To zeroth order in this parameter, the solution for the stream function is a linear distribution between two axial boundary values. First-order correction terms are calculated for a specific example.

In discussing these equations the limitations of the exact solutions due to Donaldson & Sullivan (1960) and Long (1961) are noted. These exact solutions are contrasted with the approximate treatment of this type of vortex originated by Einstein & Li (1951) and generalized by Deissler & Perlmutter (1958).

1. Introduction

An extensive historical review of analytical work on vortical flows has been given by Donaldson & Sullivan (1960). Much of this work was performed in an effort to understand the Ranque–Hilsch effect (Westley 1954). Other studies have been stimulated by vortical storms occurring in nature (Long 1958, 1961) and by concepts of advanced space propulsion (Kerrebrock & Meghreblian 1961; Rosenzweig 1961) and power generation (Lewellen 1960; McCune & Donaldson 1960) systems. The type of flow under consideration is shown in figure 1. The fluid enters tangentially with high velocity, spirals radially inward, and exits axially at some smaller radius. In practice, such a vortex may be generated in a cylindrical container or as part of an array of vortices (Rosenzweig 1961).

In spite of the large number of analytical and experimental investigations of vortices, there still exists a great deal of uncertainty concerning this complex flow pattern. Except under certain simple conditions, exact solutions of the full Navier–Stokes equations are inaccessible, so that approximate techniques must be used to interpret experimental results. Exact solutions have been given by Donaldson & Sullivan (1960), Long (1961), and Rott (1958).

An approximate treatment of vortices of this type was originated by Einstein & Li (1951) and somewhat generalized by Deissler & Perlmutter (1958). In these analyses, the axial velocity is arbitrarily taken as a discontinuous function of the radius, and has a jump at the radius of the exhaust. Continuity is then used to determine a radial velocity which is independent of the axial coordinate. The tangential velocity is assumed to be a function of the radius only and can then be determined directly from the tangential momentum equation by simple quadrature.

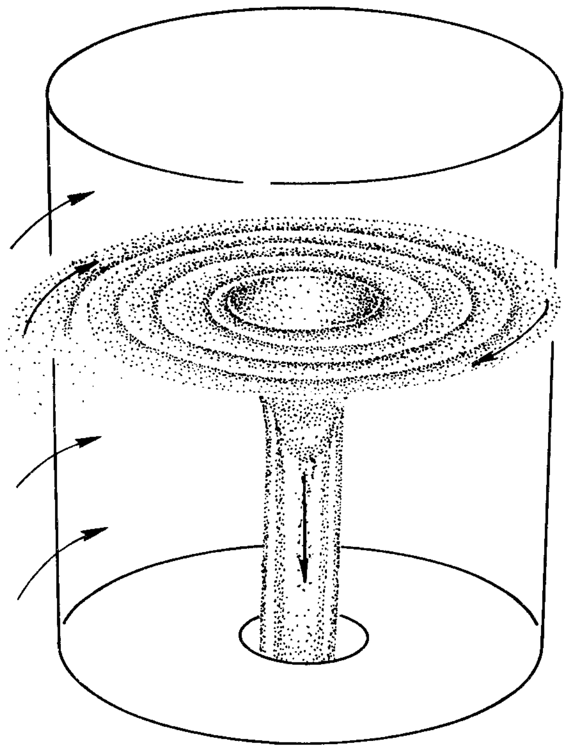

FIGURE 1. Sketch of vortex flow in which the fluid enters tangentially with high velocity, spirals radially inward and exits axially at some smaller radius.

The most serious weakness of this approach is that no consideration is given to the axial momentum equation. Actually, Donaldson & Sullivan (1960) have shown that, when the assumption is made that the radial and tangential velocities are functions of the radius only, the axial momentum equation determines the axial velocity. Unfortunately, the flows so determined cannot be made to satisfy the boundary conditions corresponding to the desired geometry. The present analysis is an attempt to clarify the relationship of the approximate solutions to the existing exact solutions and to provide a mathematically consistent approximate solution starting from the complete Navier–Stokes equations. As a result of the present study, it is shown that, when the ratio of sink flow to circulation is sufficiently small, the approximation of Einstein & Li is justified.

2. Basic equations

The equations of steady motion for an incompressible fluid with constant viscosity in cylindrical co-ordinates, assuming axial symmetry, are

$$\frac{\partial(ru)}{\partial r}+\frac{\partial(rw)}{\partial z}=0, \tag{1}$$

$$u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\frac{v^2}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}+\frac{\partial^2 u}{\partial z^2}\right), \tag{2}$$

$$u\frac{\partial v}{\partial r}+w\frac{\partial v}{\partial z}+\frac{uv}{r}=\nu\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}+\frac{\partial^2 v}{\partial z^2}\right), \tag{3}$$

$$u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}\right), \tag{4}$$

where r is the radial co-ordinate, z the axial co-ordinate, u, v and w the radial, tangential and axial components of velocity respectively, p the pressure, ρ the density and ν the kinematic viscosity.

The number of dependent variables may be reduced by defining the usual axisymmetric stream function

$$\left.\begin{aligned} u&=\frac{1}{r}\frac{\partial\hat{\psi}}{\partial z},\\ w&=-\frac{1}{r}\frac{\partial\hat{\psi}}{\partial r},\end{aligned}\right\} \tag{5}$$

and by eliminating the pressure by cross-differentiation of equations (2) and (4). The equations of motion are then

$$\frac{\partial\hat{\psi}}{\partial z}\frac{\partial\hat{\Gamma}}{\partial r}-\frac{\partial\hat{\psi}}{\partial r}\frac{\partial\hat{\Gamma}}{\partial z}=\nu\left[r^2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\hat{\Gamma}}{\partial r}\right)+r\frac{\partial^2\hat{\Gamma}}{\partial z^2}\right], \tag{6}$$

and
$$\begin{aligned} r\frac{\partial\hat{\Gamma}^2}{\partial z}=&\;3\frac{\partial\hat{\psi}}{\partial r}\frac{\partial\hat{\psi}}{\partial z}-3r\frac{\partial\hat{\psi}}{\partial z}\frac{\partial^2\hat{\psi}}{\partial r^2}+r\frac{\partial\hat{\psi}}{\partial r}\frac{\partial^2\hat{\psi}}{\partial r\partial z}+r^2\frac{\partial\hat{\psi}}{\partial z}\frac{\partial^3\hat{\psi}}{\partial r^3}-r^2\frac{\partial\hat{\psi}}{\partial r}\frac{\partial^3\hat{\psi}}{\partial r^2\partial z}\\ &-2r\frac{\partial\hat{\psi}}{\partial z}\frac{\partial^2\hat{\psi}}{\partial z^2}+r^2\frac{\partial\hat{\psi}}{\partial z}\frac{\partial^3\hat{\psi}}{\partial r\partial z^2}-r^2\frac{\partial\hat{\psi}}{\partial r}\frac{\partial^3\hat{\psi}}{\partial z^3}-\nu\left[-2r^2\frac{\partial^3\hat{\psi}}{\partial r\partial z^2}\right.\\ &\left.+2r^3\frac{\partial^4\hat{\psi}}{\partial r^2\partial z^2}+r^3\frac{\partial^4\hat{\psi}}{\partial z^4}-3\frac{\partial\hat{\psi}}{\partial r}+3r\frac{\partial^2\hat{\psi}}{\partial r^2}-2r^2\frac{\partial^3\hat{\psi}}{\partial r^3}+r^3\frac{\partial^4\hat{\psi}}{\partial r^4}\right], \end{aligned} \tag{7}$$

where
$$\hat{\Gamma}=vr. \tag{8}$$

Equations (6) and (7) can be somewhat simplified by writing the equations in terms of r^2. For convenience, the following dimensionless quantities are also introduced

$$\eta=r^2/r_0^2,\quad \xi=z/l,\quad \Gamma=\hat{\Gamma}/\Gamma_\infty,\quad \psi=\hat{\psi}/Ql, \tag{9}$$

where r_0, l, Γ_∞ and Q are appropriate dimensional quantities. For the vortex sketched in figure 1 a suitable r_0 is the radius of the exhaust hole, l the length of the vortex chamber, $2\pi\Gamma_\infty$ the circulation at the outer edge and $2\pi Q$ the volume flow per unit length.

The resulting equations are

$$\frac{\partial\psi}{\partial\xi}\frac{\partial\Gamma}{\partial\eta}-\frac{\partial\psi}{\partial\eta}\frac{\partial\Gamma}{\partial\xi}=\frac{2\eta}{N}\frac{\partial^2\Gamma}{\partial\eta^2}+\frac{\alpha}{2N}\frac{\partial^2\Gamma}{\partial\xi^2}, \tag{10}$$

and

$$\Gamma\frac{\partial\Gamma}{\partial\xi}=\epsilon\left\{4\eta^2\left[\frac{\partial\psi}{\partial\xi}\frac{\partial^3\psi}{\partial\eta^3}-\frac{\partial\psi}{\partial\eta}\frac{\partial^3\psi}{\partial\xi\partial\eta^2}-\frac{2}{N}\left(2\frac{\partial^3\psi}{\partial\eta^3}+\eta\frac{\partial^4\psi}{\partial\eta^4}\right)\right]\right.$$
$$\left.+\alpha\left[-\frac{\partial\psi}{\partial\xi}\frac{\partial^2\psi}{\partial\xi^2}+\eta\frac{\partial\psi}{\partial\xi}\frac{\partial^3\psi}{\partial\eta\partial\xi^2}-\eta\frac{\partial\psi}{\partial\eta}\frac{\partial^3\psi}{\partial\xi^3}-\frac{1}{N}\left(4\eta^2\frac{\partial^4\psi}{\partial\eta^2\partial\xi^2}+\frac{\alpha}{2}\eta\frac{\partial^4\psi}{\partial\xi^4}\right)\right]\right\}, \tag{11}$$

with $Q/\nu = N$, $(r_0/l)^2 = \alpha$ and $(Ql/\Gamma_\infty r_0)^2 = \epsilon$. Physically it is seen that the flow is governed by the three dimensionless parameters; Q/ν the radial Reynolds number; r_0/l, a ratio of characteristic lengths; and Q/Γ_∞, the ratio of volume flow per unit length to circulation.

3. Existing exact solutions

Donaldson & Sullivan (1960) have made a thorough analysis of equations (10) and (11) for the special case in which Γ is independent of z and ψ is the product of z and a function of r^2, i.e.

$$\Gamma = \Gamma(\eta). \tag{12}$$

and

$$\psi = \xi f(\eta). \tag{13}$$

For this case the equations are decoupled and yield the following equation for f:

$$ff'''-f'f''-\frac{2}{N}(2f'''+\eta f^{\text{iv}}) = 0. \tag{14}$$

With f given, Γ is determined by equation (10); thus

$$2\eta\Gamma''-Nf\Gamma' = 0. \tag{15}$$

It is immediately obvious that the restrictions placed on the flow for this case have eliminated both ϵ and α. Equation (15) is readily integrable, but, except for a few special cases [such as $f = \eta$ (Rott 1958)], (14) must be solved by numerical means. A great deal can be learned about these solutions, however, without actually solving the equations. From (5) and (13) it is seen that u is also independent of z and, thus, from equation (2):

$$\partial^2 p/\partial r\,\partial z = 0. \tag{16}$$

It is thus impossible for solutions of this type to satisfy any problem in which the axial boundary conditions force a radial variation in the axial pressure gradient. Even though the solutions presented by Donaldson & Sullivan display a very interesting cellular structure and provide considerable insight into vortex flows in general, they cannot be applied directly to most real flows occurring in nature or in the laboratory. It is necessary to know how small variations from the assumed form, (12) and (13), affect the solutions.

Long (1958) found that the equations of motion could be reduced to ordinary differential equations in terms of the similarity variable

$$\chi^* = r/z, \tag{17}$$

when $\Gamma = \Gamma(\chi^*)$ and $\psi = \xi f(\chi^*)$.

Because of the symmetry in r it might be suspected that the reduction would also work for

$$x = \left(\frac{l}{r_0}\frac{r}{z}\right)^2 = \frac{\eta}{\xi^2} \quad \text{when} \quad \Gamma = \Gamma(x) \quad \text{and} \quad \psi = \xi f(x). \tag{18}$$

With this latter transformation, (10) and (11) become

$$2x\Gamma'' - Nf\Gamma' + \alpha(3\Gamma' + 2x^2\Gamma'') = 0. \tag{19}$$

$$\begin{aligned}\Gamma\Gamma' = -2\epsilon x\Big\{ & ff''' + 3f'f'' - \frac{2}{N}(xf^{\text{iv}} + 2f''') \\ & + \frac{\alpha}{2}[2xff'''' + 6f'f'' + ff'' - 3f'^2] \\ & - \frac{\alpha}{N}[(4x + 2\alpha x^2)f^{\text{iv}} + (14x + 12\alpha x^2)f''' \\ & + (4 + \tfrac{21}{2}\alpha x)f'' - 3\alpha f']\Big\}.\end{aligned} \tag{20}$$

Equation (19) is integrable for Γ in terms of f, but when coupled with (20), numerical methods are required to solve the system. Long (1961) presented numerical solutions for this flow within a core boundary layer, i.e. $\alpha \ll 1$.

Without actually solving these equations, it can be shown that a particular feature of these solutions is that the axial velocities are of the same order as the tangential velocities even at large radii (Long 1958). Furthermore, with α small and $|N|$ of order one or larger, ϵ is restricted to values of order one. This last statement can be seen by noting that for any real flow, Γ must go to zero on the axis, and thus Γ' is of order one. Consequently, for f and its derivatives to be of order one, (20) requires that ϵ be of order one, i.e. Q/Γ_∞ and r_0/l are of the same order. The case of interest here, that of radial sink flow with strong circulation which turns and exhausts axially near the centre line, is thus not included in this class of solutions.

4. New expansion procedure

Examining (11), it can be seen that even a very small change in Γ with ξ could effect a profound change on ψ if the parameter $(Ql/\Gamma_\infty r_0)^2$ is equally small. For many physical flows of interest, $Q/\Gamma_\infty < 10^{-2}$, so that even for moderately large l/r_0, ϵ is small. Equation (11) thus suggests a series expansion of Γ and ψ in ϵ with the leading term of Γ independent of the axial co-ordinate.

Formally, let us assume

$$\Gamma = \sum_{n=0}^{\infty} \Gamma_n(\eta, \xi)\,\epsilon^n, \tag{21}$$

$$\psi = \sum_{n=0}^{\infty} \psi_n(\eta, \xi)\,\epsilon^n. \tag{22}$$

Substituting into (10) and (11) and equating coefficients of the powers of ϵ, yields for ϵ^0 from (11)

$$\Gamma_0 \partial\Gamma_0/\partial\xi = 0. \tag{23}$$

Therefore
$$\Gamma_0 = \Gamma_0(\eta), \tag{24}$$
and from (10), using (24):
$$2\eta\Gamma_0'' - N\Gamma_0'\partial\psi_0/\partial\xi = 0. \tag{25}$$
It follows that
$$\psi_0 = f_{00}(\eta) + \xi f_{01}(\eta), \tag{26}$$
and
$$2\eta\Gamma_0'' - Nf_{01}\Gamma_0' = 0. \tag{27}$$

To this order of approximation then, any f_{01} can be assumed and Γ_0 found by integrating (27). This is essentially what is done in generating the approximate solutions of Einstein & Li (1951) and Deissler & Perlmutter (1958). By judiciously choosing f_{01} a wide range of boundary conditions can be satisfied. The crucial question is, how accurate are the resulting solutions? It might be expected that the error involved is of order ϵ, since this is the order of terms neglected in the equations of motion, but this is true only if the assumed series for Γ and ψ in ϵ are convergent in some sense.

The next higher order set of equations, found by equating coefficients of ϵ, is

$$\Gamma_0\Gamma_{12} = 2\eta^2\left[f_{01}f_{01}''' - f_{01}'f_{01}'' - \frac{2}{N}(2f_{01}''' + \eta f_{01}^{\text{iv}})\right], \tag{28}$$

$$\Gamma_0\Gamma_{11} = 4\eta^2\left[f_{01}f_{01}''' - f_{01}'f_{01}'' - \frac{2}{N}(2f_{00}''' + \eta f_{00}^{\text{iv}})\right], \tag{29}$$

$$2\eta\Gamma_{12}'' - Nf_{01}\Gamma_{12}' - 3Nf_{13}\Gamma_0' + 2Nf_{01}'\Gamma_{12} = 0, \tag{30}$$

$$2\eta\Gamma_{11}'' - Nf_{01}\Gamma_{11}' - 2Nf_{12}\Gamma_0' + 2Nf_{00}'\Gamma_{12} + Nf_{01}'\Gamma_{11} = 0, \tag{31}$$

$$2\eta\Gamma_{10}'' - Nf_{01}\Gamma_{10}' - Nf_{11}\Gamma_0' + Nf_{00}'\Gamma_{11} + \alpha\Gamma_{12} = 0, \tag{32}$$

with
$$\Gamma_1 = \Gamma_{10}(\eta) + \xi\Gamma_{11}(\eta) + \xi^2\Gamma_{12}(\eta), \tag{33}$$

$$\psi_1 = f_{10}(\eta) + \xi f_{11}(\eta) + \xi^2 f_{12}(\eta) + \xi^3 f_{13}(\eta). \tag{34}$$

Equations (27) through (32) provide a set of 6 equations with 9 unknowns.† This freedom in the truncated series can be used to advantage to provide the desired freedom in the axial boundary conditions. That is, the set can be made complete by supplying information at one or more axial positions in the form of boundary conditions.

Possibly the most interesting problem of this nature is the boundary-value problem for the stream function, that is, to have given

$$\psi(\eta, \xi_0) = g_0(\eta) \tag{35}$$
and
$$\psi(\eta, \xi_1) = g_1(\eta). \tag{36}$$

By a suitable choice of co-ordinate axis and the dimensional parameter l, one can set $\xi_0 = 0$ and $\xi_1 = 1$ with no loss in generality. If these boundary conditions are independent of ϵ, they correspond to the following equations:

$$f_{00}(\eta) = g_0(\eta), \tag{37}$$

$$f_{01}(\eta) = g_1(\eta) - g_0(\eta), \tag{38}$$

$$f_{n0}(\eta) = 0, \quad n > 0, \tag{39}$$

$$\sum_{k=1}^{2n+1} f_{nk}(\eta) = 0 \quad (n > 0). \tag{40}$$

† The possibility of terminating these series in such a way as to obtain an exact solution is discussed by Lewellen (1962).

Equations (37)–(40) provide exactly the right number of equations to make the set of equations obtained to any specified order of ϵ a complete set.

If $O(\epsilon)$ and higher are ignored the system of equations reduces to (27), (37) and (38). That is, one is justified in specifying the stream function and merely solving for the corresponding circulation profile, as is done by Einstein & Li, and Deissler & Perlmutter. Physically, this implies that, in flows of vanishingly small ϵ, the stream function is completely specified by its boundary conditions. It is merely a linear extrapolation between two given axial boundary values since the equation of motion to be satisfied by it is of higher order in ϵ.

At the next level of approximation, that is keeping terms of order ϵ while neglecting those of higher order, the system of equations is composed of (27)–(32), (37), (38), and from (39) and (40) the two additional equations

$$f_{11}+f_{12}+f_{13} = 0, \tag{41}$$

$$f_{10} = 0. \tag{42}$$

This forms a complete set of 10 equations and 10 unknowns. However, the equations are all decoupled and can therefore be solved one at a time. The only unusual feature of the system is that most of the equations call for differentiation rather than integration. The equations are solved in the following sequence: (37), (38), and (42) give f_{00}, f_{01} and f_{10} directly; equation (27) is then integrated to give Γ_{01}. Formally, the integration of (27) leads to

$$\Gamma_0 = \frac{\int_0^\eta \left[\exp \tfrac{1}{2}N \int_0^t \eta^{-1} f_{01}(\eta)\, d\eta\right] dt}{\int_0^\infty \left[\exp \tfrac{1}{2}N \int_0^t \eta^{-1} f_{01}(\eta)\, d\eta\right] dt}, \tag{43}$$

with $\Gamma_0(\infty) = 1$. (The boundary conditions in η will be discussed later.) The term Γ_{12} is then found from (28) by differentiating f_{01}. Similarly, Γ_{11}, f_{13} and f_{12} are found from (29), (30), and (31) respectively by differentiating known functions. Now f_{11} can be determined by (41), and (32) integrated to yield Γ_{10}. The integral for Γ_{10} in terms of the other variables is

$$\begin{aligned}\Gamma_{10} = \int_0^\eta \left[\phi(t)\int_0^t \frac{N(\Gamma_0'(x) f_{11}(x) + \Gamma_{11}(x) f_{00}'(x))}{2x\phi(x)}\, dx\right] dt - A\int_0^\eta \phi(t)\, dt \\ -\alpha\left\{\int_0^\eta \left[\phi(t)\int_0^t \frac{\Gamma_{12}(x)}{2x\phi(x)}\, dx\right] dt - B\int_0^\eta \phi(t)\, dt\right\},\end{aligned} \tag{44}$$

with
$$\phi(t) = \left[\exp \tfrac{1}{2}N \int_0^t \eta^{-1} f_{01}(\eta)\, d\eta\right],$$

$$A = \left[\int_0^\infty \phi(\eta)\, d\eta\right]^{-1} \int_0^\infty \left[\phi(t)\int_0^t \frac{N(\Gamma_0'(\eta) f_{11}(\eta) + \Gamma_{11}(\eta) f_{00}'(\eta))}{2\eta\phi(\eta)}\, d\eta\right] dt,$$

and
$$B = \left[\int_0^\infty \phi(\eta)\, d\eta\right]^{-1} \int_0^\infty \left[\phi(t)\int_0^t \frac{\Gamma_{12}(\eta)\, d\eta}{2\eta\phi(\eta)}\right] dt.$$

Some care must be exercised in carrying through this process since the solution involves several differentiations of the prescribed boundary conditions, g_0 and g_1.

Clearly, from (11) these functions must be differentiable at least 6 times, but an even more stringent requirement is that the boundary values of Γ_{11}, Γ_{12}, f_{13} and f_{12} at $\eta = 0$ and $\eta = \infty$ must derive from the form of g_0 and g_1.

To continue the discussion, it is necessary to examine the boundary conditions at $\eta = 0$ and $\eta = \infty$. From the definitions of ψ and Γ, it is obvious that, in order to keep velocities finite on the axis, it is necessary to have

$$\left.\frac{\partial\psi}{\partial z}\right|_{r=0} = \left.\frac{\partial\psi}{\partial r}\right|_{r=0} = \Gamma(0, z) = 0. \tag{45}$$

The boundary conditions at the outer radius of any region are somewhat more arbitrary. The conditions imposed by Donaldson & Sullivan were

$$\Gamma(r_0, z) = \text{const.}, \quad \left.\frac{\partial\psi}{\partial z}\right|_{r=r_0} = \text{const.}, \quad \left.\frac{\partial\psi}{\partial r}\right|_{r=r_0} = 0. \tag{46}$$

To these conditions is added the condition

$$\left.\frac{\partial p}{\partial z}\right|_{r=r_0} = 0, \tag{47}$$

since this is the condition which could not be specified by Donaldson & Sullivan without arriving at the trivial solution $\psi = 0$, as has been discussed earlier. These conditions correspond to the flows which possess a rather strong axial pressure gradient along the central axis which tapers off asymptotically to zero at some larger radius. This represents flows that consist of circulation imposed upon radial sink flow at large radius which turn to stagnation flow near the centreline, i.e. flow in a container in which the fluid is forced in uniformly at some radius and removed through a hole of smaller radius at one end (see figure 1). These conditions might also approximate some flows in nature such as tornadoes, dust devils and waterspouts.

From (45)–(47) the corresponding conditions on Γ_{nk} and f_{nk} can be derived. Applying conditions at large radius in an asymptotic sense, they are:

$$\left.\begin{aligned} \Gamma_{nk}(0) &= f_{nk}(0) = 0 \quad (\text{all } n, k), \\ \Gamma_0(\infty) &= 1 = f_{01}(\infty) \quad (f_{00}(\infty) = 0), \\ \Gamma_{nk}(\infty) &= f_{nk}(\infty) = 0 \quad (n \neq 0). \end{aligned}\right\} \tag{48}$$

Returning to the discussion of the restrictions on the form of g_0 and g_1, it can be seen that the conditions imposed by (48) on Γ_{11}, Γ_{12}, f_{13} and f_{12} must be inherent in the form of g_0 and g_1. It can be shown that the boundary conditions at $\eta = 0$ are satisfied if g_0 and g_1 are analytic functions of η in some neighbourhood of $\eta = 0$ and

$$f_{01}(0) = g_1(0) - g_0(0) = 0. \tag{49}$$

At the other end, as $\eta \to \infty$, a form which satisfies (48) and has sufficient generality for most purposes is

$$g_0 \sim \eta^{m_0} e^{-\eta}, \quad g_1 \sim 1 + \eta^{m_1} e^{-\eta} \tag{50}$$

with m_0 and m_1 arbitrary, since all multiples of the higher derivatives of these functions go to zero at infinity.

To be more specific let us consider an example in which the solution will be carried through to $O(\epsilon)$. To simplify the problem somewhat consider a plane of symmetry at $\xi = 0$. This immediately implies that $f_{00} = g_0 = 0$. Also from (29) and (31), $\Gamma_{11} = f_{12} = 0$. Consider the stream function at $\xi = 1$ to be specified as $g_1 = 1 - e^{-\eta}$ and the boundary conditions at $r = \infty$ and $r = 0$ to be those given in (48). The flow which closely approximates the flow sketched in figure 1, if the effect of surface boundary layers is neglected, is now completely determined.

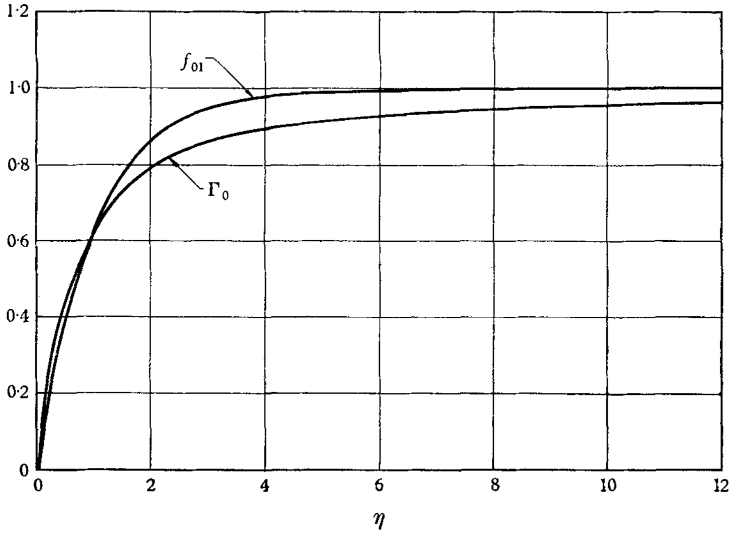

FIGURE 2. Zeroth-order stream function and circulation as functions of η for $N = -4$ and the axial boundary conditions of $g_0 = 0$ and $g_1 = 1 - e^{-\eta}$.

$$\Gamma_0 = \frac{\int_0^{\eta}\left[\exp \tfrac{1}{2}N\int_0^t \eta^{-1}f_{01}\,d\eta\right]dt}{\int_0^{\infty}\left[\exp \tfrac{1}{2}N\int_0^t \eta^{-1}f_{01}\,d\eta\right]dt}; \quad f_{01} = (1-e^{-\eta}).$$

Equation (38) yields f_{01}, (43) yields Γ_0, (28) Γ_{12}, (30) f_{13}, (41) f_{11} and (44) yields Γ_{10}. These functions are plotted in figures 2 to 4. From (44) the direct relationship of α to Γ_{10} can be indicated by writing $\Gamma_{10} = \Gamma_{100} + \alpha\Gamma_{101}$.

Of the three dimensionless flow variables, α, N and ϵ, only the N-dependence remains entangled in the functions. The functional dependence on α and ϵ has been factored out so that figures 2 to 4 are valid for arbitrary α and ϵ. Of course, it must be remembered that this solution neglects $O(\epsilon^2)$ terms in the equations of motion. $N = -4$ was chosen for this computation as a representative value of the radial Reynolds number.

The example just cited could be carried to higher order in ϵ, but not without considerable numerical effort. It can be shown (Lewellen 1962) that such a procedure does indeed lead to a series which is asymptotically convergent in the sense that an ϵ can be found which makes the ratio of the $(n+1)$th term to the nth term as small as desired for any finite n.

An interesting facet of this proof is the manner in which N affects the convergence of the series. It is seen that, as N becomes large negatively, the convergence conditions upon ϵ become more restrictive. That is, as the flow becomes more nearly inviscid, the circulation equation tends to decouple from that of the stream function. The circulation is a constant at large radius independent

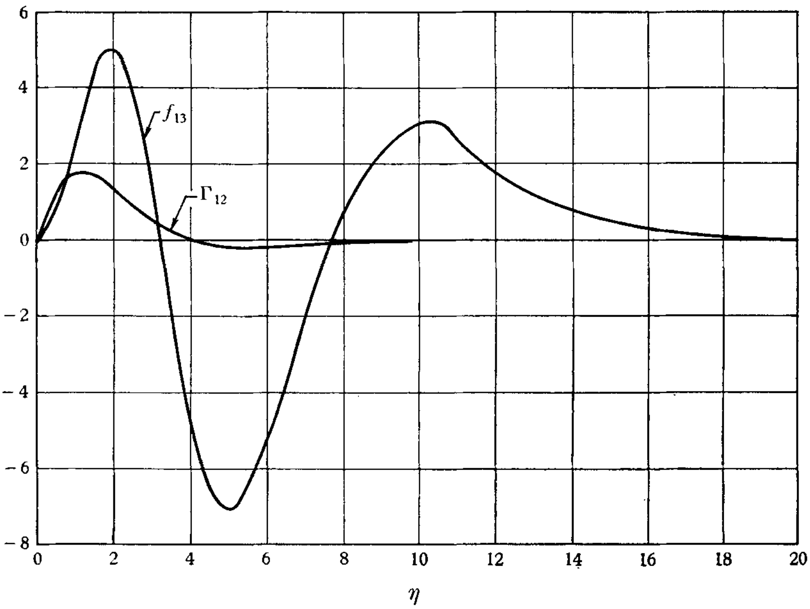

FIGURE 3. f_1 and Γ_1 as functions of η for $N=-4$, $g_0=0$ and $g_1=1-e^{-\eta}$.

$$f_{13}=\frac{1}{3N\Gamma_0'}\{2\eta\Gamma_{12}''+2Nf_{01}'\Gamma_{12}-Nf_0\Gamma_{12}'\};$$
$$\Gamma_{12}=\frac{1}{\Gamma_0}\left[\frac{4}{N}\eta^3+2\eta^2\left(1-\frac{4}{N}\right)\right]e^{-\eta}.$$

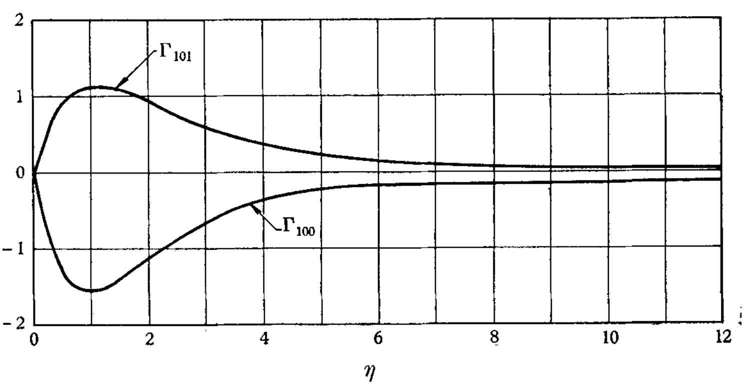

FIGURE 4. $\Gamma_{10}=\Gamma_{100}+\alpha\Gamma_{101}$ as a function of η for $N=-4$, $g_0=0$ and $g_1=1-e^{-\eta}$.

of the axial dimension, and as the viscosity becomes smaller the circulation tends to remain constant to smaller radii. But as long as there is no radial variation, there can be no axial variation and thus from (11) the circulation can have no effect on the stream function, unless ϵ approaches zero. This series approach thus requires that $(Ql/\Gamma_\infty r_0)$ approaches zero more rapidly than (Q/ν) approaches negative infinity.

It should be noted that to obtain (37) to (40) it was assumed that the boundary conditions, (35) and (36), were independent of ϵ. If this is not true it is necessary to know g_0 and g_1, as functions of ϵ, to the same order that it is desired to determine ψ and Γ. It is physically reasonable that the solution be determined only to the same degree to which the boundary conditions are specified.

5. Axial boundary conditions

The problem posed in the introduction is to find a vortex solution which will permit certain boundary conditions to be satisfied, i.e. the problem of interest is a boundary-value problem. Although other conditions may be imposed to define a problem which could be solved in a similar manner, the boundary-value problem on the stream function has been chosen here as the most interesting, physically. For this problem it is necessary to specify the stream function at two axial positions as well as at two radial positions.

In the exact solutions of Donaldson & Sullivan, and also that of Long, the question of axial boundary conditions does not arise. This is a direct consequence of the restrictive requirements imposed upon the flow in both cases. They find particular modes of the axial velocity which satisfy the equations of motion and certain radial boundary conditions. Donaldson & Sullivan give the modes which are of the form $zf(r)$ (with $\Gamma = \Gamma(r)$) and Long deals with those which are of the form $(1/r)f(r/z)$ (with $\Gamma = \Gamma(r/z)$). These special forms are not the only possibilities. As can be readily checked in (10) and (11), taking $\Gamma = z\Gamma(r)$ would lead to still more modes of w, of the form $zf_1(r)+f_2(r)$. In addition, within the present series approach, it is possible to find first-order similar solutions [i.e. solutions which are similar neglecting order ϵ^2 terms (Lewellen 1962)]. Persons with ingenuity and persistence might find other particular modes which satisfy the equations of motion. However, due to the non-linearity of the equations, these modes cannot have the same importance as they enjoy in linear analyses. There is no way of using these particular functions to build up solutions to fit general boundary conditions. The functions can only be used individually and, therefore, do not permit any freedom in the axial boundary conditions.

In contrast, the present method provides a general solution capable of satisfying axial boundary conditions which possess certain analytical forms. By the same token this method only determines the solution uniquely to the same degree to which these boundary conditions are uniquely specified.

The question of what conditions are physically imposed in any particular problem is not as straightforward as it might first appear. This difficulty is caused by the fact that the present method cannot be carried into the boundary layers usually found in the immediate neighbourhood of physical axial boundaries. Within these thin regions, Γ must go from order one at the outside to zero at the inside (unless the boundary itself is rotating). From this and the boundary-layer equations, it can be deduced that $u/v = O(1)$ within this boundary layer (Mack 1962), thus invalidating the series solution in powers of Q/Γ_∞. The boundary condition upon the present solution must be applied outside these boundary layers. But the usual hydrodynamic approximation of ignoring the influence of the boundary-layer flow on the outer flow is very poor for this particular problem since what might be termed secondary flow induced by the boundary layer can dramatically change the boundary conditions and hence produce a major effect on the outside flow.

A demonstration of this phenomenon is shown in figure 5, plate 1. Ink has been introduced into a water vortex near the bottom wall with a small hypodermic tube. The ink is carried into the end-wall boundary layer and then ejected upward near the $\frac{1}{8}$th in. diameter exhaust hole at the centre. Some of the ink reaches the top where it interacts with the top end-wall boundary layer before it returns to the bottom to be exhausted out of the box. The flow is quite different from the simple picture which would exist without the boundary layers.

Of course, the boundary-layer flow in turn depends upon the characteristics of the outer flow. This problem requires a delicate matching of the flows inside and outside the boundary layer. It is necessary to have the boundary-layer solution correct to the same order of ϵ to which it is desired to find the solution in the main body of the flow.

Ignoring the inherent difficulties presented by the boundary layers, i.e. assuming properly rotating boundaries, there is still some ambiguity concerning the exhaust hole. Rotation of the flow forces strong radial variations of the pressure even within the exhaust radius. Depending upon the relation of the pressure inside the vortex to that outside the exhaust there may be flow sucked into the vortex along the axis from outside. Rather than prescribing a stream function as a boundary condition, it would seem appropriate here to apply some type of integral condition on the total pressure over this area. More work is needed to clarify the influence of the ambient pressure at the exhaust on the flow pattern within the vortex.

6. Conclusions

A solution of the axisymmetric, three-dimensional vortex flow pattern has been found in the form of an asymptotic series for small $\epsilon = (Ql/\Gamma_\infty r_0)^2$. To zeroth order in ϵ the solution for the stream function is simply a linear distribution between two axial boundary values. With this stream function, two quadratures yield the zeroth order circulation, which is independent of the axial co-ordinate. Thus to zeroth order in ϵ the family of solutions considered by Donaldson & Sullivan has been extended to include not only their cases of constant axial pressure but almost any radial variation of axial pressure gradient.

The method for carrying the solution to higher order has been described and an illustrative example computed through first-order terms.

The accuracy of the solution is principally limited by the knowledge of the axial boundary conditions on the stream function. In most physical occurrences these boundary conditions must be found by solving both a boundary-layer problem on the surfaces and the recirculation problem of the exhaust.

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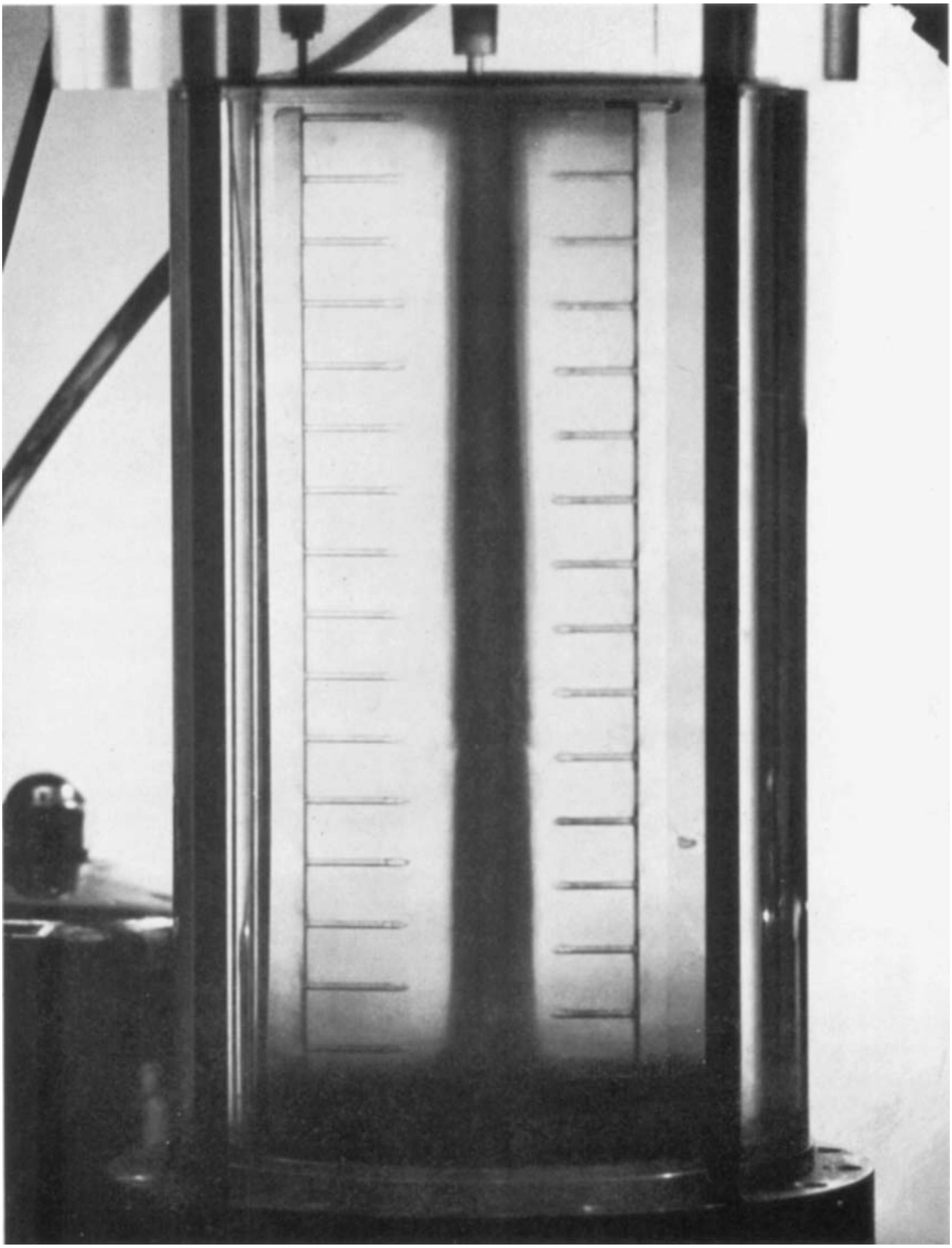

FIGURE 5, PLATE 1. Ink injected into a water vortex near the bottom wall demonstrating the dramatic effect on the flow produced by the boundary layer.